

Renormalized 2PN spin contributions to the accumulated orbital phase for LISA sources

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We give here a new third post-Newtonian (3PN) spin-spin contribution (in the PN parameter ε) to the accumulated orbital phase of a compact binary, arising from the spin-orbit precessional motion of the spins. In the equal mass case this contribution vanishes, but LISA sources of merging supermassive binary black holes have typically a mass ratio of 1:10. For such non-equal masses this 3PN correction is periodic in time, with period approximately ε^{-1} times larger than the period of gravitational waves. We derive a *renormalized* and simpler expression of the spin-spin coefficient at 2PN, as an average over the time-scale of this period of the combined 2PN and 3PN contribution. We also find that for LISA sources the quadrupole-monopole contribution to the phase dominates over the spin-spin contribution, while the self-spin contribution is negligible even for the dominant spin. Finally we define a renormalized total spin coefficient $\bar{\sigma}$ to be employed in the search for gravitational waves emitted by LISA sources.

I. INTRODUCTION

Gravitational waves are generated by time-varying quadrupolar (or higher multipolar) mass (energy) configurations in general relativity. Although a convincing indirect proof for the existence of the theoretically predicted gravitational waves is provided by binary pulsars [1], the most well-known being the Hulse-Taylor pulsar [2] (others are also known [3],[4]) a major effort is undergoing nowadays for the first direct detection of gravitational waves by the most sensitive Earth-based interferometric gravitational wave detectors LIGO [5] and VIRGO [6].

These advanced instruments are able to detect in the frequency range 10 – 1000 Hz. Among their most likely sources are compact binary systems composed of neutron stars / stellar size black holes. In order to simplify the route towards the first detection, some of the relevant astrophysical details of the system are commonly omitted, like the spins and quadrupole moments. The detection effort concentrates on the signal emitted by point masses on quasi-circular orbit. With this, only the mass ratio should be monitored, while the inclusion of the astrophysical characteristics would induce many new parameters (additional six, for example, by the two spins). However the ambitious future aim of a gravitational wave based astrophysics would certainly require to include the spins and quadrupole moments of the binary components in the data analysis.

Supermassive black holes (SMBHs) of mass $3 \times 10^6 \div 3 \times 10^9$ solar masses are known to exist in the center of the majority of galaxies [7], [8], [9], including ours. When galaxies merge, and the separation of the SMBHs is of a few parsecs, the SMBH binary enters a dissipative regime dominated by dynamical friction (an interaction of the SMBHs with the already merged stellar environment). As the relative separation further shrinks, this is followed by the gravitational radiation dominated dissipative regime, starting at approximately one hundredth of a parsec.

During dynamical friction some of the orbital angular momentum of the binary black hole system is transferred to the stellar environment. In the process the stellar population at the poles of the system is ejected and a torus is formed [10], [11]. There had been a major worry, that the loss-cone mechanism for feeding stars into orbits that intersect the binary black holes is too slow and the SMBHs will stall in their approach to each other (the last parsec problem; for a review see [12]). However recently more than one mechanism was proposed, which lead to a further approach of the SMBHs: (1) direct interaction with the surrounding stars slightly further outside [11]; (2) relaxation processes due to cloud / star - star interactions [13], both repopulating the stellar orbits in the center of the galaxy. In a series of papers it has been recently shown that (3) even in the absence of two-body relaxation or gas dynamical processes, unequal mass and / or eccentric binaries with the mass larger than 10^5 solar masses can shrink to the gravitational wave emission regime of one hundredth of a parsec in less than a Hubble time due to the binary orbital decay by three-body interactions in the gravitationally-bound stellar cusps [14]. Alternatively, by considering (4) the three accretion disks: one around each black hole and a third one, which is circumbinary, it has been shown in [15] that the balance of the orbital angular momentum changing processes (the circumbinary disk removes orbital angular momentum from the binary via the binary-disk resonant interaction, however the mass transfer to each individual black hole adds orbital angular momentum to the binary) is such that these binaries will merge within a Hubble time driven by this mechanism.

In consequence by one mechanism or another the SMBHs would be able to approach each other to distances smaller than approximately one hundredth parsecs, when the gravitational radiation becomes the dominant dissipative effect. The gravitational waves in the frequency range $10^{-4} - 10^{-1}$ Hz emitted by such a system of coalescing SMBHs will be captured by the forthcoming LISA space mission [16].

There are a few important comments to make on these SMBH LISA sources.

(a1) The mass distribution of the galactic central SMBHs shows a broken powerlaw [17], [18], confirmed by most recent work [19] and a recent observational survey [20].

(a2) Let $\nu = m_2/m_1$ be the mass ratio of the binary with masses $m_2 \leq m_1$. Thus $0 \leq \nu \leq 1$. Next we define $m = m_1 + m_2$ as the total mass, and $\mu = m_1 m_2 / m$ as the reduced mass, ranging between ν for $\nu \rightarrow 0$ to $1/4$ for equal masses. Further we introduce the symmetric mass ratio $\eta = \mu/m = (2 + \nu + \nu^{-1})^{-1}$ (for small mass ratios $\eta \rightarrow \nu$). The probability for a specific mass ratio for SMBH encounters can be derived as an integral over the black hole mass distribution, folded with the rate to actually merge (proportional to the capture cross section and the relative velocity for two galaxies). By this method it has been estimated in [21] that the most likely mass ratios are in the range $\nu \in (1/30, 1/3)$. Therefore the most typical mass ratio for LISA sources would be $\nu = 10^{-1}$, to be considered in this paper.

(b1) The spin of individual black holes has a quite high value. As the surrounding matter is accreted into the hole, the angular momentum transfer spins up the black hole, close to the maximally allowed value. By taking into account the angular momentum transfer from accreting matter alone, the dimensionless spin parameter $\chi = S/m^2$ (in units $c = 1 = G$) grows to the maximally allowed value of 1 even if the initial state of the black hole was non-rotating [22]. By taking into account the energy input of the in-falling (horizon-crossing) photons emitted from the steady-state thin accretion disk, a torque counteracting to the one due to mass accretion has been found [23], reducing the limiting value of the dimensionless spin parameter to 0.9982. The description of the process can be further enhanced by (i) including the contribution of the open or closed magnetic field lines in the vicinity of the disk [24], (ii) discussing a symbiotic system (introduced in [25]) of a rotating black hole, a magnetized thin accretion disk and jets in the magnetosphere of the hole [26]. With any of these refinements, the end state of the black hole will stay very close to the extremal Kerr limit.

(b2) In consequence the spin of the individual components of a binary black hole system, which is formed typically by capture events (even for stellar mass black holes the emergence of binary systems by stellar evolution is quite unlikely) will be also high.¹

We conclude from these that any reasonable analysis of the gravitational waves produced by central galactic SMBH mergers should take into account both the individual spins S_i of the black holes and their typical mass ratio.

In the inspiral phase of the merger, which can be described by post-Newtonian (PN) techniques and lasts until the innermost stable orbit (ISO) is reached, the spins and quadrupole moments induce a quasi-precessional evolution of the spin vectors [31]. The leading contributions come from the SO precessions at 1.5PN order. The interaction of the individual spins lead to a 2PN spin-spin (SS) quasi-precessional contribution to the dynamics. Similar 2PN quasi-precessional contributions arise, when one binary component is regarded as a monopole moving in the quadrupolar field of the other (quadrupole-monopole, QM contribution). This angular evolution of the spins is decoupled from the radial motion, which was completely integrated in terms of a generalized Kepler equation [32]. The inclusion of the angular evolution (spin precessions) into the description besides increasing the accuracy, also breaks many degeneracies among parameters, greatly improving (relative to previous estimates) how well LISA will be able to measure the masses, the locations of binaries on the sky, and the distance to the source [33]. The gravitational waveforms for spinning compact binaries with precession effects through 1.5PN order were also recently analyzed using spin-weighted -2 spherical harmonics, and applied to spinning, non-precessing binaries [34].

A careful and detailed discussion, following the assumptions of Ref. [35], by taking into account both the SO precession and the dissipative effect of gravitational radiation (averaged over one radial period) for the above typical mass ratio has already resulted in two achievements [21]. The first is a quantitative description of the spin-flip process, visible in the X-shaped radio galaxies. Secondly, the precessional phase of the merger of two black holes, occurring prior to the spin-flip, was suggested to be the source of the superdisk, visible in radio galaxies. The precessing jet acts as a super-wind separating the two radio lobes in the final stages of the merger. According to this model such radio galaxies are candidates for subsequent SMBH mergers [21].

¹ How large is the final spin after the merger occurs, is still the object of debate. Recent numerical work on the merger of two black holes with spin has predicted (beside recoil effects arising for particular configurations [27], interesting in their own), the final spin state of the merged black hole. Extrapolating from the particular configurations analyzed, an analytical formula was also proposed for the final spin [28]. This formula has been employed in the analysis of the final spin as arising from a combination of accretion and merger sequence [29]. Alternative analyses of the final spin in mergers have been performed by analytical or numerical methods, yielding various predictions for the values of the final spin [30].

The actual detection of gravitational waves requires the precise knowledge of the waveforms. This is determined both by the relative geometry of the sources and detectors, and by the evolution of the sources. In the latter, the accumulated orbital phase ϕ plays a central role, as any effect that causes the template to differ from the actual signal by one cycle over the 500 to 16,000 accumulated cycles in the sensitive bandwidth will result in a substantial reduction in the signal-to-noise ratio [36]. Therefore the phase of the gravitational waves emitted by a compact binary is required to high PN accuracy. Various contributions up to 3.5 PN orders are known. Among these, there are important spin and mass quadrupole contributions, like SO at 1.5PN, also SS, QM and SS-self contributions at 2PN. (The latter arise from the self-interaction of the spins [37].) We note here, that alternative definitions of the phase in the spinning case were introduced (the orbital phase with respect to the ascending node [38]) and the rigorous waveform containing SO contributions has been also computed [39].

Besides possible quadrupolar irregularities in the mass distribution of the binary components (which are however quite unlikely due to the strong self-gravity of these objects yielding rapidly to a spherical configuration), and the mutual quadrupolar deformation of them (shown to be quite small in [40]), the proper rotation is the effect generating a considerable quadrupole moment for each black hole. The quadrupolar deformation can be characterized by the coefficients $p_i = Q_i/m_i m^2$, with Q_i the quadrupole scalar. If the quadrupole moment originates entirely in the rotation of the compact object (what we shall assume in what follows), then $Q_i \simeq -\alpha \chi_i^2 m_i^3$, with the parameter $\alpha \in (4, 8)$ for neutron stars, depending on their equation of state. Stiffer equations of state give larger values of α , while $\alpha = 1$ for rotating black holes [41]. Here $\chi_i = S_i/m_i^2$ are the dimensionless rotation parameters and the negative sign is because the rotating compact object is centrifugally flattened, becoming an oblate spheroid.

In what follows, we will consider maximally rotating SMBHs ($\chi_i \approx 1$, thus $S_i \approx m_i^2$), with their quadrupole moment entirely determined by the rotation. Therefore $p_i = -S_i^2/m_i^2 m^2 \approx -(m_i/m)^2$, thus

$$p_1 \approx -(1+\nu)^{-2}, \quad (1)$$

$$p_2 \approx -(1+\nu^{-1})^{-2}. \quad (2)$$

The ratio of the spins of the compact objects (characterized by $R_i \approx m_i$ and evaluated for comparable and high rotation velocities $V_1 \approx V_2 \approx 1$ according to our assumption of maximal rotation) can be expressed as

$$\frac{S_2}{S_1} \approx \left(\frac{m_2}{m_1}\right)^2 = \nu^2, \quad (3)$$

while in the same approximation the ratio of the spins with the magnitude L of orbital angular momentum \mathbf{L} becomes

$$\frac{S_2}{L} \approx \frac{m_2^2 V_2}{\mu r v} = \left(\frac{m}{r}\right) \left(\frac{1}{v}\right) V_2 \frac{m_2}{m_1} \approx \varepsilon^{1/2} \nu, \quad (4)$$

$$\frac{S_1}{L} = \frac{S_2}{L} \frac{S_1}{S_2} \approx \varepsilon^{1/2} \nu^{-1}, \quad (5)$$

where we have employed the definition of the PN parameter $\varepsilon \approx m/r \approx v^2$. As remarked in [21], in case of a small $\nu \approx \eta$ the ratio S_2/L is shifted towards even smaller values (therefore $S_2 \ll L$ during all stages of the inspiral), however the ratio of the spin of the dominant compact object to the magnitude of the orbital angular momentum can change drastically. Indeed, it is determined by the relative magnitude of the small parameters ε and ν . As ε increases during the inspiral, whenever ν falls in the range of $\varepsilon^{1/2}$, an $S_1 \approx L$ epoch is reached, which follows the initial epoch with $S_1 < L$ and precedes the forthcoming $S_1 > L$ epoch. This happens for the mass ratios $\nu \in (1/30, 1/3)$.

We also note that in previous works we have found useful to choose as some of the parameters for the new spin degrees of freedom the relative angles κ_i between the spins and orbital angular momentum and the angle γ between the spins. This is because both the quasi-precessional equations for the spins and the equations describing the gravitational radiation losses up to 4.5 PN orders close in these variables, the energy E and magnitude of the orbital angular momentum (averaged over one radial orbit) \bar{L} [42], [43], [44], [45].

In this paper we will review the contributions induced by the spins and quadrupole moments to the accumulated orbital phase of the binary and we propose modifications in this description. In order to do so, in Section II we review the existing contributions up to 3.5PN orders. Then we represent graphically the dependence of the QM contribution on the parameter space (κ_1, κ_2) . We illustrate that due to the involved numerical coefficients, the self-spin contribution, although of 2PN order, is much smaller, than the QM contribution, thus it can be safely dropped at this accuracy. Both the QM and the SS-self contributions are discussed for either equal or unequal masses. The dependence of the SS contribution on the parameter space (κ_1, κ_2) and on the relative spin configuration, expressed in terms of the relative spin azimuthal angle $\Delta\psi$ is discussed only in the equal mass case here. (Due to a spherical triangle identity this variable is connected to the previously mentioned angles κ_i and γ .)

In Section III we discuss the consequences of the leading order precession effects on these coefficients. We show that due to the SO precession the SS contribution to the phase receives 3PN time-dependent corrections, which happen to cancel in the equal mass case. At non-equal masses however these contributions vary periodically, with a period T_{3PNSS} which is ε^{-1} times longer than the period of the gravitational wave T_{wave} .

Then in Section IV we look for LISA sources of typical mass ratio $\nu = 10^{-1}$. We represent here the detailed time evolution of the total spin-spin coefficient on the time scale T_{3PNSS} in the parameter space (κ_1, κ_2) . Next we repeat this for the sum of SS and QM contributions, showing that the QM effects dominate and the SS effects modulate. We are then able to define a renormalized SS coefficient at 2PN, which can be considered constant up to 3PN on a time-scale $T_{3PNSS} = \varepsilon^{-1} T_{wave}$. This coefficient is simpler than its unnormalized counterpart.

In Section V we compare the number of cycles left for typical LISA sources, considering the following cases: (a) the 2PN accurate spin-spin coefficient, previously employed in the literature, with the relative spin angles γ, κ_i being constants at this accuracy (the precessions contributing only with higher order terms, thus dropped here); (b) our renormalized spin-spin coefficient (constant up to 3PN), without the precession equations, which are already taken care of to this accuracy by our renormalization procedure; (c) the numerical evolution, with time-varying spin-spin coefficient, found by including the precessions.

Finally in the Concluding Remarks we summarize our findings and define the renormalized value of the total coefficient encompassing both the QM and SS contributions, which due to its simplicity, time-independence up to 3PN, and much better agreement with the numerical results is more suitable to be used in future LISA data analysis than the presently available expression.

Throughout the paper we use units $G = 1 = c$.

II. THE PHASE OF GRAVITATIONAL WAVES TO 3.5 PN ACCURACY

The integrated orbital phase of gravitational waves has been computed to high accuracy, with contributions both from general relativistic corrections to the Keplerian motion and other Newtonian contributions due to the finite-size of the compact binary components. The respective contributions were recently summarized in [46], [37], and [47]. Up to the desired 3.5 PN accuracy the phase can be formally decomposed as:

$$\phi = \phi_c + \phi_N + \phi_{1PN} + \phi_{1.5PN} + \phi_{2PN} + \phi_{2.5PN} + \phi_{3PN} + \phi_{3.5PN} , \quad (6)$$

where ϕ_c is a constant, and the lower order contributions are completely known as:

$$\phi_N = -\frac{1}{\eta} \tau^{5/8} , \quad (7)$$

$$\phi_{1PN} = -\frac{1}{\eta} \left(\frac{3715}{8064} + \frac{55}{96} \eta \right) \tau^{3/8} , \quad (8)$$

$$\phi_{1.5PN} = -\frac{3}{4\eta} \left(\frac{1}{4} \beta - \pi \right) \tau^{1/4} , \quad (9)$$

$$\phi_{2PN} = -\frac{1}{\eta} \left(\frac{9275495}{14450688} + \frac{284875}{258048} \eta + \frac{1855}{2048} \eta^2 - \frac{15}{64} \sigma \right) \tau^{1/8} , \quad (10)$$

$$\phi_{2.5PN} = -\frac{1}{\eta} \left[\left(-\frac{38645}{172032} + \frac{65}{2048} \eta \right) \pi + \beta_{PN} \right] \ln \tau . \quad (11)$$

Here the dimensionless time parameter $\tau = \eta(t_c - t)/5m$ is decreasing until the merger. The contribution at 1PN is purely relativistic, computed in [48]. At 1.5 PN appear the leading-order tail contributions [49]-[52], together with the spin-orbit contributions [54], [55], encompassed in the coefficient

$$\beta = \frac{1}{12} \sum_{i=1}^2 \frac{S_i \cos \kappa_i}{m_i^2} \left(113 \frac{m_i^2}{m^2} + 75\eta \right) . \quad (12)$$

At 2PN there are both relativistic contributions [53] and finite size effects, collected into the coefficient

$$\sigma = \sigma_{S_1 S_2} + \sigma_{SS-self} + \sigma_{QM} , \quad (13)$$

$$\sigma_{S_1 S_2} = \frac{S_1 S_2}{48 \eta m^4} (-247 \cos \gamma + 721 \cos \kappa_1 \cos \kappa_2) , \quad (14)$$

$$\sigma_{SS-self} = \frac{1}{96 m^2} \sum_{i=1}^2 \left(\frac{S_i}{m_i} \right)^2 (6 + \sin^2 \kappa_i) , \quad (15)$$

$$\sigma_{QM} = -\frac{5}{2} \sum_{i=1}^2 p_i (3 \cos^2 \kappa_i - 1) , \quad (16)$$

The enlisted contributions originate in the spin-spin [54], [55], self-spin [37] and quadrupole-monopole [41] interactions.

The dependence of the spin coefficients σ_{QM} and $\sigma_{SS-self}$ on the relative angles κ_i is represented both for equal masses and $\nu = 10^{-1}$ on Fig. 1, under the assumptions of maximal rotation and quadrupole moment generated purely by rotation. Although formally both contributions are of 2PN order, the numerical values of the coefficients involved imply that the quadrupole-monopole contribution clearly dominates over the self-spin contribution, which only takes values in the narrow interval $\sigma_{SS-self} \in (1/32, 7/96) \sum_{i=1,2} (m_i/m)^2$.

The coefficient σ_{QM} strongly depends on the spin configuration, in particular in the $\nu = 10^{-1}$ case only on the relative angle between the dominant spin and orbital angular momentum. By comparing the two panels of the figure, we see that the σ_{QM} contribution increases considerably with decreasing mass ratio.

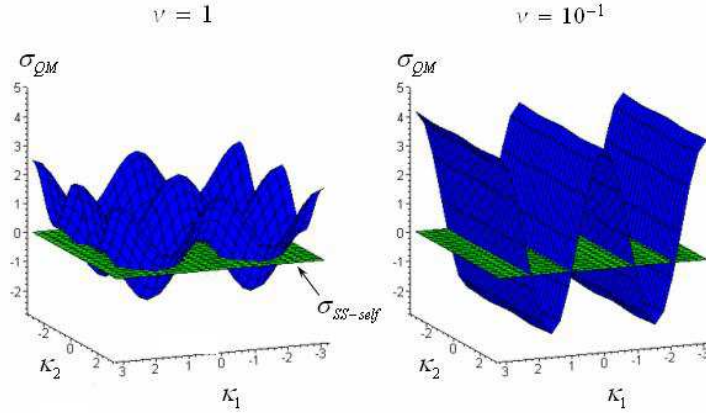


FIG. 1: (Color online). The quadrupole-monopole contribution σ_{QM} (blue wavy surface) represented for equal masses and $\nu = 10^{-1}$ in the parameter space (κ_1, κ_2) . The coefficient σ_{QM} strongly depends on the spin configuration, in particular in the $\nu = 10^{-1}$ case only on the relative angle between the dominant spin and orbital angular momentum. The σ_{QM} contribution increases considerably with decreasing mass ratio. For comparison, the much smaller self-spin contribution $\sigma_{SS-self}$ is also represented (green levelled surface). The values of $\sigma_{SS-self}$ are positive, but well below 1. The plots are for maximally spinning binaries, with the quadrupole moment induced by rotation.

In the *equal mass case* we also show the dependence of $\sigma_{S_1 S_2}$ and of the total σ on the parameter space (κ_1, κ_2) for various relative spin azimuthal angles $\Delta\psi$ (Figs. 2 and 3). Here $\Delta\psi = \psi_2 - \psi_1$ is the difference between the azimuthal angles ψ_i of the spin vectors. As the vertical scale for all these figures is the same, we can clearly see that the quadrupole-monopole contribution dominates for a huge part of the parameter space. Also, in general $\sigma \neq 0$ and its positive values are higher than the negative ones due to the QM contribution. However there is a small set of particular configurations for which the contributions to σ cancel out each other.

The coefficients $\sigma_{S_1 S_2}$ and σ in the *non-equal mass case* will be discussed in the remaining Sections of the paper. They are more complicated because (anticipating the results of the next Section) the relative spin azimuthal angle $\Delta\psi$ becomes time-dependent for $\nu \neq 1$ due to precessions.

Finally at 2.5 PN there are contributions to ϕ from the first PN correction to SO [56], [47], expressed by

$$\beta_{PN} = \sum_{i=1}^2 \frac{S_i \cos \kappa_i}{m_i^2} \left[\left(\frac{965}{3584} \eta + \frac{681145}{516096} \right) \frac{m_i^2}{m^2} + \left(\frac{37265}{57344} + \frac{1735}{7168} \eta \right) \eta \right] , \quad (17)$$

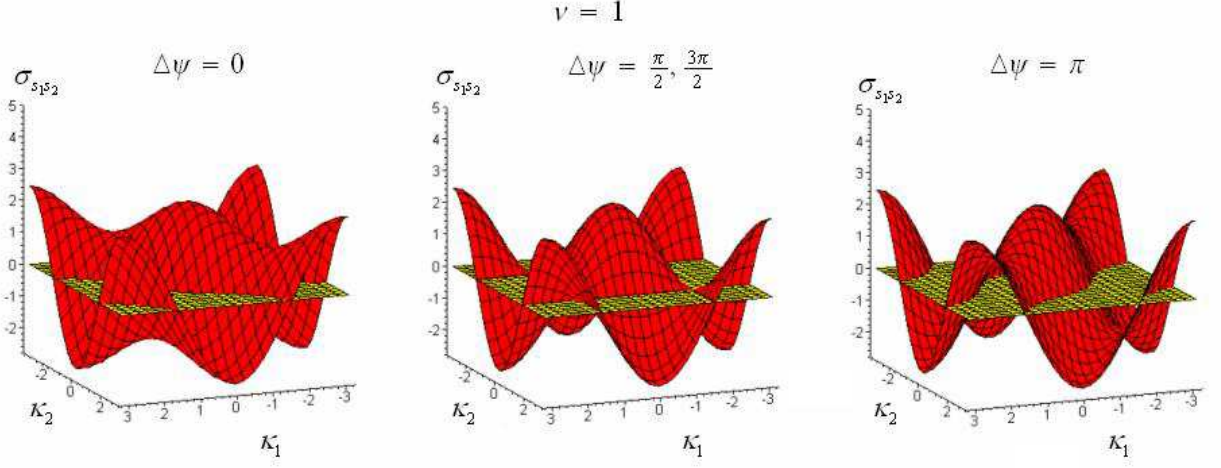


FIG. 2: (Color online). The dependence of the spin-spin parameter $\sigma_{s_1 s_2}$ (red wavy surfaces) on the angles κ_i for equal masses and various values of the relative spin azimuthal angle $\Delta\psi = 0, \pi/2, \pi, 3\pi/2$. For comparison the zero level surface is also represented. Plot for the maximal rotation case.

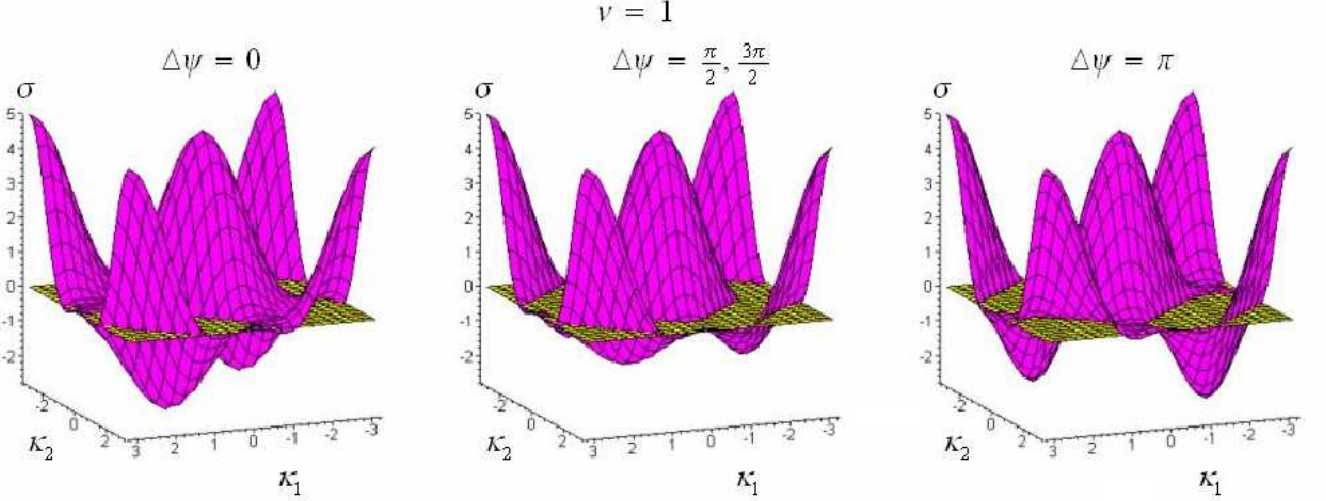


FIG. 3: (Color online). Same as on Fig 2, but for the total coefficient σ (lilac wavy surfaces). Although there are particular configurations (at the intersection with the zero level surfaces) for which the various contributions to σ cancel out each other, for the vast majority of the parameter space $\sigma \neq 0$. The positive values are typically higher than the negative ones due to the QM contribution.

and tail effects [57]. We note that the logarithmic terms arising from the latter can be eliminated through a modified definition of the phase [58]

The list of contributions known at 3PN and 3.5PN are not exhaustive. At 3PN there are known relativistic and tail contributions [46]

$$\begin{aligned}
 \phi_{3PN} = & -\frac{1}{\eta} \left\{ \frac{831032450749357}{57682522275840} - \frac{53}{40}\pi^2 - \frac{107}{56}C \right. \\
 & + \frac{107}{448} \ln\left(\frac{\tau}{256}\right) + \left[-\frac{126510089885}{4161798144} + \frac{2255}{2048}\pi^2 \right] \eta \\
 & \left. + \frac{154565}{1835008}\eta^2 - \frac{1179625}{1769472}\eta^3 \right\} \tau^{-1/8}, \quad (18)
 \end{aligned}$$

where $C = \lim_{n \rightarrow \infty} [\sum_{k=1}^n (1/k) - \ln n] \approx 0.577$ is the Euler-Mascheroni constant². However the contributions from the PN corrections of the SS, self and QM contributions are not known.

At 3.5PN only the tail contribution has been calculated [46]

$$\phi_{3.5PN} = -\frac{1}{\eta} \left(\frac{188516689}{173408256} + \frac{488825}{516096} \eta - \frac{141769}{516096} \eta^2 \right) \pi \tau^{-1/4}, \quad (19)$$

but clearly there are numerous other contributions, as the 2PN correction to the SO contribution, SO-SS, SO-QM couplings etc.

It is the main purpose of this paper to add new spin-spin contributions at 3PN, arising from the spin-orbit precession of the spin vectors and discuss their implication to the definition of an effective σ at 2PN.

III. LEADING ORDER PRECESSIONAL CONTRIBUTION TO σ

The spin vectors undergo a complicated quasi-precessional motion [31], and the angles κ_i and γ evolve accordingly. In Ref. [42] it was derived that the leading order contributions to the quasi-precessional variation of the angles κ_i is of $\varepsilon^{3/2}$ order, while γ evolves in the ε order. Supplementing the SO and SS contributions already considered in Ref. [42] with the QM contributions (with the quadrupole moment arising from pure rotation), by employing Eqs. (1)-(5) and the method described in a footnote of Ref. [44], according to which $O(\delta x) = O(\dot{x})r\varepsilon^{-1/2}$, we have checked that these estimates are not changed, although the leading order changes in κ_i acquire QM type contributions. However the leading order change in γ arises solely from the SO precession.

With the angles κ_i and γ non-constant, the coefficients β , σ and β_{PN} also evolve in time. From among the relative spin angles the contributions σ_{QM} and $\sigma_{SS-self}$ contain only κ_i . As these start to vary at $\varepsilon^{3/2}$ order, the variations of κ_i would contribute only at 3.5PN orders. The same is true for the κ_i -dependence of $\sigma_{S_1 S_2}$. However the variation at ε order of γ induces 3PN corrections to $\sigma_{S_1 S_2}$. Thus the leading order precessional contribution to σ is caused by the leading order precessional evolution of the spin angle γ , of SO origin.

Therefore in what follows we derive the time dependence of γ . From the SO precession of the spin vectors (the SO contribution in Eq. (2.17c) of Ref. [42], with the change of notation $\eta \rightarrow \nu$) we find

$$(\cos \gamma)^{\bullet} = -\frac{3L(\nu - \nu^{-1})}{2r^3} \sin \kappa_1 \sin \kappa_2 \sin \Delta\psi. \quad (20)$$

By taking the time derivative of the spherical cosine identity

$$\cos \gamma = \cos \kappa_1 \cos \kappa_2 + \cos \Delta\psi \sin \kappa_1 \sin \kappa_2. \quad (21)$$

we get

$$\begin{aligned} \sin \gamma \frac{d\gamma}{dt} &= \sin \kappa_1 \sin \kappa_2 \sin \Delta\psi \frac{d}{dt} \Delta\psi \\ &\quad + \cos \Delta\psi \left[\cos \kappa_1 \frac{d}{dt} (\cos \kappa_2) + \cos \kappa_2 \frac{d}{dt} (\cos \kappa_1) \right]. \end{aligned} \quad (22)$$

The second term can be dropped as we need to consider this relation only to ε accuracy. A comparison of Eqs. (20) and (22) gives:

$$\frac{d}{dt} \Delta\psi = \frac{3L_N(\nu^{-1} - \nu)}{2r^3}, \quad (23)$$

whenever $\kappa_i, \Delta\psi \neq 0$. For equal masses $\Delta\psi = \text{const}$, therefore γ is also unchanged by precessional effects to ε order. Therefore $\sigma_{S_1 S_2}$ can be considered constant even at 3PN for $\nu = 1$.

For any $\nu \neq 1$ Eq. (23) can be integrated in the following way. First we note that in order to calculate the leading order change in $\Delta\psi$, all quantities are needed only to Keplerian order. We also specialize to circular orbits $r = a$, as

² The Euler-Mascheroni constant appears, among other places, in expressions involving the exponential integral, the Laplace transform of the natural logarithm, an inequality for Euler's totient function and solution of the second kind to Bessel's equation [59].

required by the derived expression of the phase. We then pass to integration in terms of the true anomaly χ , defined as

$$\frac{d\chi}{dt} = \frac{L}{\mu a^2} , \quad (24)$$

and employ the Kepler equation to establish the connection between the time and the eccentric anomaly ξ (which to leading order and circular orbits coincides with the true anomaly χ):

$$n(t - t_0) = \chi , \quad (25)$$

where $n = 2\pi/T_{orbit}$. By choosing $t_0 = 0$, we obtain the required explicit time dependence of γ :

$$\cos \gamma = \cos \kappa_1 \cos \kappa_2 + \cos \left[(\Delta\psi)_0 + \frac{3\mu n (\nu^{-1} - \nu)}{2a} t \right] \sin \kappa_1 \sin \kappa_2 , \quad (26)$$

which gives a new time-dependent 3PN spin contribution to $\sigma_{S_1 S_2}$.

The angle γ (and the respective 3PN contribution to $\sigma_{S_1 S_2}$) has a periodicity with period

$$T_{3PNSS} = \frac{4\pi a}{3\mu (\nu^{-1} - \nu) n} = \frac{2a}{3\mu (\nu^{-1} - \nu)} T_{orbit} . \quad (27)$$

With the approximation valid for quasicircular orbits: $T_{orbit} = 2T_{wave}$, also employing $\varepsilon \approx m/a$ and $\eta^{-1} = m/\mu = (2 + \nu + \nu^{-1})$, we get:

$$\frac{T_{3PNSS}}{T_{wave}} = \frac{4}{3(\nu^{-1} - \nu)\varepsilon\eta} = \frac{4(2 + \nu + \nu^{-1})}{3(\nu^{-1} - \nu)\varepsilon} . \quad (28)$$

It has been shown in [21], that the typical mass ratio at galactic black hole encounters is between 1/3 to 1/30, such that the mass ratio $\nu = 10^{-1}$ can be considered fairly representative. We adopt this value in what follows. Then

$$\frac{T_{3PNSS}}{T_{wave}} \approx 1.63 \varepsilon^{-1} . \quad (29)$$

We have therefore found that for the representative mass ratio the time-scale of variation of the angle γ , (and the related change in $\sigma_{S_1 S_2}$) is of ε^{-1} times larger order, than the period of gravitational waves.

IV. THE TIME-EVOLVING $\sigma_{S_1 S_2}$ AND σ IN THE UNEQUAL MASS CASE $\nu = 10^{-1}$

We remark on Eq. (26) that on a time-scale T_{3PNSS} , the time-dependent contribution averages out:

$$\overline{\cos \gamma} = \cos \kappa_1 \cos \kappa_2 , \quad (30)$$

therefore the average of the spin coefficient $\sigma_{S_1 S_2}$ becomes

$$\overline{\sigma_{S_1 S_2}} = \frac{79 S_1 S_2}{8 \eta m^4} \cos \kappa_1 \cos \kappa_2 . \quad (31)$$

We call this the *renormalized* spin-spin coefficient in the phase of gravitational waves. A comparison with the standard expression (14) shows that the renormalized expression is simpler than the original one.

The spin coefficient $\sigma_{S_1 S_2}$ to 3PN accuracy can be then rewritten as a sum of the renormalized coefficient and a time-varying part:

$$\sigma_{S_1 S_2} = \overline{\sigma_{S_1 S_2}} + \delta\sigma , \quad (32)$$

$$\delta\sigma = -\frac{247 S_1 S_2}{48 \eta m^4} \sin \kappa_1 \sin \kappa_2 \cos \left[(\Delta\psi)_0 + \frac{3\mu n (\nu^{-1} - \nu)}{2a} t \right] . \quad (33)$$

The second term is periodic (with period $T_{3PNSS} \approx \varepsilon^{-1} T_{wave}$ for the typical binaries with mass ratio $\nu = 10^{-1}$), and as such, it averages out over a time scale $\varepsilon^{-1} T_{wave}$.

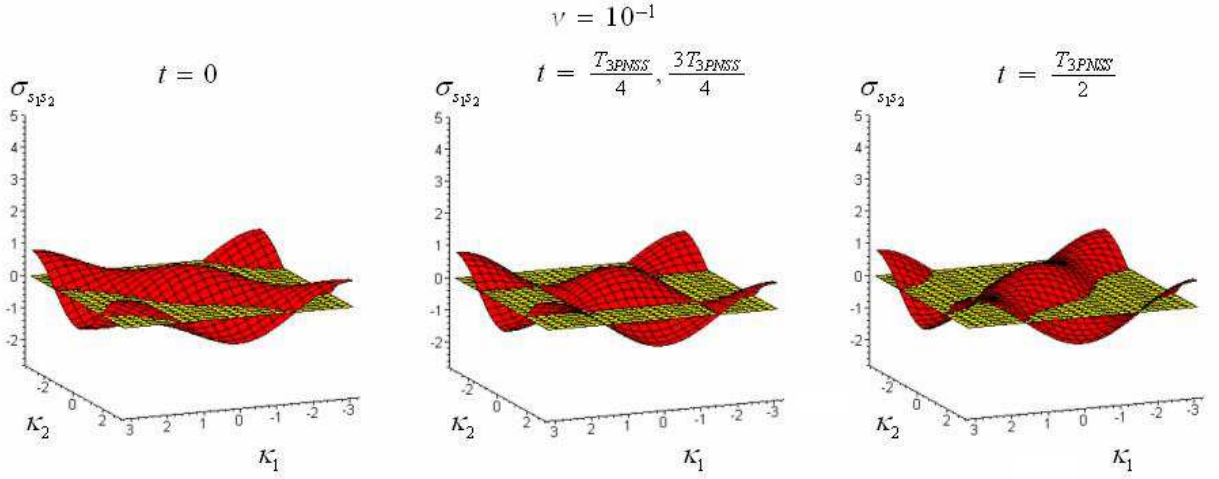


FIG. 4: (Color online). The time-evolving spin-spin parameter $\sigma_{S_1 S_2}$ for mass ratio $\nu = 10^{-1}$ represented every quarter period during T_{3PNSS} .

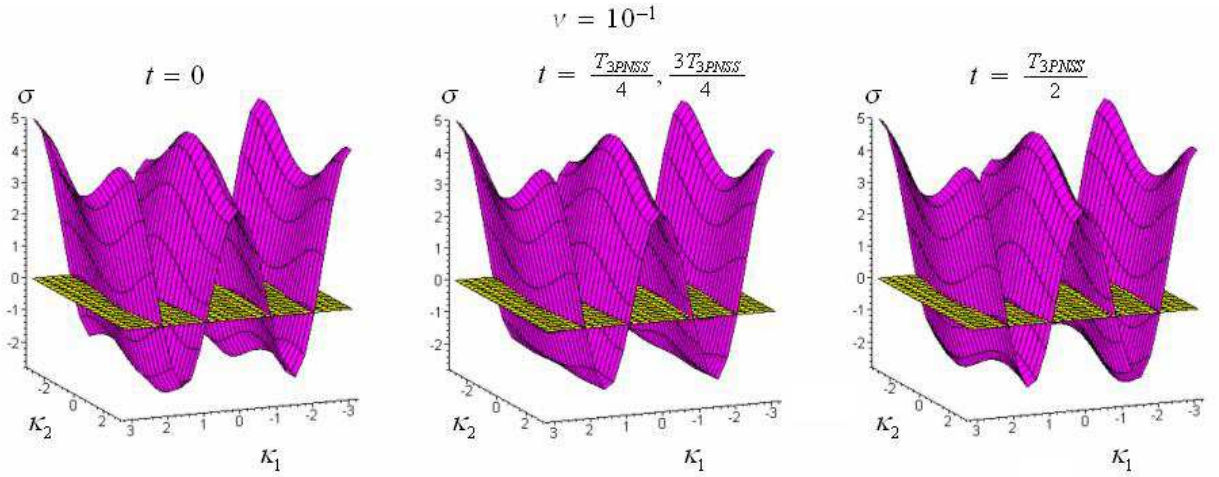


FIG. 5: (Color online). The time-evolving parameter σ for mass ratio $\nu = 10^{-1}$ represented every quarter period during T_{3PNSS} . One can see that the dominant contribution is from the quadrupole-monopole interaction.

On Figs 4 and 5 we have represented the time-evolving parameters $\sigma_{S_1 S_2}$ and σ for the mass ratio $\nu = 10^{-1}$ in a time sequence sampled every quarter period during T_{3PNSS} .³ While the importance of the spin-spin contribution decreased as compared to the equal mass case (the scales on Figs 2 and 4 are the same for easy comparison), the quadrupole-monopole contribution is enhanced by a small ν (Fig 1). As a result, for LISA sources the dominant contribution comes from the quadrupole-monopole interaction and the time-varying spin-spin contribution only induces a wobbling of the crests determined by the quadrupole-monopole interaction (compare Figs 1 and 5).

³ Animated gif-s showing the temporal evolution of both $\sigma_{S_1 S_2}$ and σ are available at [60].

V. ANALYTICAL AND NUMERICAL CONTRIBUTIONS TO THE ACCUMULATED NUMBER OF GRAVITATIONAL WAVE CYCLES

The phase of the gravitational waves (6) emerges by two integrations over time from the expression of the time derivative of the orbital angular frequency $\dot{\omega}$, see Ref. [37]. The spin coefficient $\sigma_{S_1 S_2}$ first appears in the latter expression and to a 2PN accuracy it can be considered constant, hence it also emerges in the phase.

To a higher accuracy the coefficient acquires time-dependence from the precession equations, which should change the results of the integration. Alternatively, the contribution of the leading order SO precession can be taken into account by the averaging procedure described in the previous section. Then the spin coefficient $\overline{\sigma_{S_1 S_2}}$, regarded as a constant, formally leads to the same expression of the phase and accumulated number of gravitational wave cycles $\mathcal{N} = (\phi_c - \phi) / \pi$ as Eq. (12) of Ref. [37], with the renormalized coefficient $\overline{\sigma_{S_1 S_2}}$ replacing $\sigma_{S_1 S_2}$. Obviously this result is approximate rather than exact. The question comes, how good is the approximation?

As the exact time integrations turn out to be cumbersome and the result is not expressible in terms of elementary functions, in order to answer this question we calculate numerically the corresponding contribution to the accumulated number of gravitational wave cycles. We proceed in the following way. Eq. (7) of Ref. [37] gives the evolution of the orbital angular frequency ω in time to 2PN accuracy. By inserting the time dependent expression (32) into this equation, and keeping only the Newtonian and the time-dependent correction of the spin-spin terms, we obtain:

$$\frac{d(m\omega)}{d\tau} = -96 (m\omega)^{11/3} - 96 (m\omega)^5 \delta\sigma. \quad (34)$$

The solution of this equation can be also split into the known Newtonian contribution and a term of $\delta\sigma$ order:

$$m\omega(\tau) = \frac{\tau^{-3/8}}{8} + m\omega_{\delta\sigma}(\tau). \quad (35)$$

Here the dimensionless time τ is defined as $\tau = \eta(t_c - t) / 5m$, with $(t - t_c)$ the time until the coalescence. (Note that we are discussing only the precessional corrections to the spin-spin contribution in this paper, without attempting to encompass other cumbersome post-Newtonian corrections of the spin-spin effects arising from the dynamics.)

Rather than giving the number of cycles left in terms of τ , we express it as an integral over the orbital angular frequency

$$\mathcal{N} = \mathcal{N}_N + \mathcal{N}_{\delta\sigma} = \frac{1}{\pi} \int_{\tau_i}^{\tau_f} \omega(\tau) d\tau = \frac{1}{\pi} \int_{\omega_i}^{\omega_f} \frac{\omega}{\dot{\omega}} d\omega. \quad (36)$$

Here $\dot{\omega}$ is given by Eq. (34). In order to express $\delta\sigma(\omega)$, from Eq. (33) by employing Kepler's third law $\omega^2 a^3 = m$ and $n = \omega$ (valid to leading order), we obtain

$$T_{3PNSS} = \frac{4\pi m}{3\eta(\nu^{-1} - \nu)(m\omega)^{5/3}}. \quad (37)$$

We also express t as function of ω to leading order from Eq. (35) as

$$t - t_c = -\frac{5m}{2^8 \eta} (m\omega)^{-8/3}. \quad (38)$$

Thus

$$\delta\sigma = -\frac{247 S_1 S_2}{48 \eta m^4} \sin \kappa_1 \sin \kappa_2 \cos \left[(\Delta\psi)_0 + \frac{3\eta(\nu^{-1} - \nu) t_c}{2m} (m\omega)^{5/3} - \frac{15(\nu^{-1} - \nu)}{2^9} (m\omega)^{-1} \right]. \quad (39)$$

Inserting the expression (34), with $\delta\sigma$ given by Eq. (39) into Eq. (36) we obtain the number of cycles in the frequency range (ω_i, ω_f) . This enables us to give accurate numerical results for the accumulated number of cycles.

For comparison we select three LISA sources with unequal mass given in Tables I and II of Ref. [62] and we enlist in Table I the corresponding spin-spin contributions calculated for particular (constant) values of the angles κ_i in three different ways: (a) by employing Eq. (10) of Ref. [37]; (b) in the same way, but with our renormalized spin-spin coefficient $\overline{\sigma_{S_1 S_2}}$ replacing $\sigma_{S_1 S_2}$; (c) employing a numerical integration based on the exact method described in this section. For comparison we also give the corresponding Newtonian, PN and QM contributions. The data confirms that for these non-equal mass binaries the QM contribution dominates over the spin-spin contribution. It is also immediate to see that the renormalized spin-spin coefficient leads to much closer values to the results of the numerical integration, then the 2PN-accurate coefficient.

TABLE I: Newtonian, PN, QM and spin-spin contributions to the accumulated number of gravitational wave cycles for three LISA sources with unequal mass, for particular values of the angles κ_i and $(\Delta\psi)_0$. The relevant frequency ranges (with $f = \omega/2\pi$) are also indicated. The spin-spin contributions are computed in three distinct ways (a) in a 2PN rigorous formalism ($\mathcal{N}_{\sigma_{SS}}$); (b) with the renormalized spin-spin coefficient ($\mathcal{N}_{\bar{\sigma}_{SS}}$); and (c) taking into account the exact precessional evolutions up to 3PN by numerical integration ($\mathcal{N}_{\bar{\sigma}_{SS}+\delta\sigma_{SS}(t)}$). The numbers obtained with the renormalized coefficient are within 0.02 cycles agreement with the numerical data, presenting an improvement up to 0.6 cycles over the 2PN accurate expression. The numbers in the table confirm that for unequal masses the QM contribution dominates over the spin-spin contribution.

$(\Delta\psi)_0 = 15^\circ$ $\kappa_1 = \kappa_2 = 30^\circ$	$1.4M_\odot$ $400M_\odot$	10^4M_\odot 10^5M_\odot	10^6M_\odot 10^7M_\odot
$f_{in}(Hz)$	4.601×10^{-2}	4.199×10^{-4}	2.361×10^{-5}
$f_{fin}(Hz)$	1.000	3.997×10^{-2}	3.997×10^{-4}
\mathcal{N}_N	2294624.602	21055.705	1174.257
\mathcal{N}_{PN}	35363.312	677.247	114.536
$\mathcal{N}_{\sigma_{QM}}$	-432.606	-13.712	-6.028
$\mathcal{N}_{\sigma_{SS}}$	-2.986	-2.678	-1.177
$\mathcal{N}_{\bar{\sigma}_{SS}}$	-3.588	-3.218	-1.415
$\mathcal{N}_{\bar{\sigma}_{SS}+\delta\sigma_{SS}(t)}$	-3.588	-3.207	-1.404

VI. CONCLUDING REMARKS

We have discussed the contributions induced by the spin-spin and quadrupole-monopole coupling to the accumulated orbital phase. We have shown that the contributions from the self-coupling of the spins are much smaller than the QM and SS contributions in all cases (Fig 1), therefore they can be dropped at a 2PN accuracy. The QM contributions depend on the parameter space (κ_1, κ_2) , while the SS contributions on $(\kappa_1, \kappa_2, \Delta\psi)$ [or alternatively on $(\kappa_1, \kappa_2, \gamma)$], and were represented on these parameter spaces.

For equal mass compact binaries we have shown that the QM contributions (left panel of Fig 1) and the SS contributions (Fig 2) are of comparable size, although the SS contributions are slightly higher. These two contributions (at least for positive values) happen to roughly amplify each other, thus their combined contribution (Fig 3) has approximately twice as large positive amplitude, than the individual contributions. There is also a set of measure zero in the parameter space, for which the total contribution vanishes.

For typical LISA sources with unequal masses $\nu = 10^{-1}$ the SS contribution becomes time-dependent due to the variation of the angle γ by SO precession. The time-evolution of the SS coefficient was shown on Fig 4. By comparing with the QM contribution for $\nu = 10^{-1}$ (right panel of Fig 1), we see that the SS contribution is much smaller. As consequence, when we add these contributions, the combined coefficient shows the pattern of the QM contribution, modulated in time by the SS contribution (Fig 5).

The time variation of the SS component was shown to be periodic, with a period ϵ^{-1} times longer than the period of the gravitational wave. By separating from the SS contribution a time-varying component which averages to zero on this time-scale, we may call the rest as a renormalized SS coefficient $\bar{\sigma}_{S_1S_2}$. Besides of being constant up to 3PN, it has a simpler expression, than the previously known 2PN SS coefficient $\sigma_{S_1S_2}$.

Therefore we find useful to introduce in the unequal mass case the total renormalized coefficient $\bar{\sigma} = \sigma_{QM} + \bar{\sigma}_{S_1S_2} + \sigma_{SS-self}$ for maximally rotating black holes with quadrupole moment due to rotation. It has the expression

$$\bar{\sigma} \approx \frac{5}{2} \left[3 \cos^2 \kappa_1 - 1 + \nu \left(2 + \frac{79}{20} \cos \kappa_1 \cos \kappa_2 - 6 \cos^2 \kappa_1 \right) \right], \quad (40)$$

which was obtained by keeping only the leading and first order QM and SS contributions in ν . The renormalized $\bar{\sigma}$ is constant in time up to 3PN and as such is more convenient to use in wave detection, than σ , which is constant only to 2PN. For all numerical examples studied, the renormalized coefficient gave results within 0.02 cycle accuracy as compared to the numerical results. This represented an improvement of $0.4 \div 0.6$ cycles over the previously employed 2PN accurate coefficient.

The dependence of $\bar{\sigma}$ on the parameter space (κ_1, κ_2) is shown on Fig 6. It is very similar to the pure quadrupole-monopole effect, represented in the second panel of Fig 1, which is slightly modulated by the renormalized spin-spin contribution.

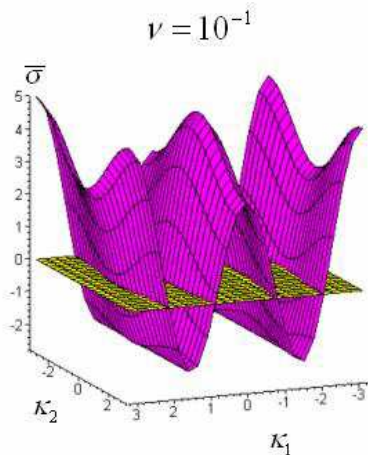


FIG. 6: (Color online). The renormalized parameter $\bar{\sigma}$ for mass ratio $\nu = 10^{-1}$. The dominant contribution comes from the quadrupole-monopole interaction (compare with the second panel of Fig 1), which is slightly modulated by the renormalized spin-spin contribution.

Due to the arguments presented in this paper we propose to replace σ by $\bar{\sigma}$ in the expression of the gravitational wave phase, Eq. (10), whenever it is applied for LISA sources.

In a similar way to the time dependence and renormalization of σ , a time dependence and renormalization of the spin-orbit contribution β can be derived, the calculation of which involves to integrate the combined SO, SS and QM quasi-precessional time evolution of the angles κ_i at $\varepsilon^{3/2}$ order. This has been done for the equal mass case [61], however it seems to be far from obvious for $\nu \neq 1$. Therefore we defer the study of this effect to a future work.

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